

Absolute purity in motivic homotopy theory

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joint work with F. Déglise, J. Fasel and A. Khan

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The absolute purity conjecture

Grothendieck's **absolute (cohomological) purity conjecture** (SGA5, Expose I 3.1.4) is the following statement: if $i : Z \rightarrow X$ is a closed immersion between noetherian regular schemes of pure codimension c , $n \in \mathcal{O}(X)$ and $\mathcal{O}_Z = \mathbb{Z} = n\mathbb{Z}$, then the étale cohomology sheaf supported in Z with values in \mathbb{Z} can be computed as

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This conjecture has been solved by Gabber.

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Based on Thomason's method + rigidity for algebraic K -theory

Importance of the absolute purity conjecture

The absolute purity property, together with resolution of singularities, is frequently used in cohomological studies of schemes:

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Motivic homotopy theory

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Absolute purity in motivic homotopy theory

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- Study the coniveau spectral sequence.

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- Main result: the absolute purity in motivic homotopy theory is satisfied with rational coefficients in mixed characteristic.

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Motivic homotopy theory

- The motivic homotopy theory or \mathbb{A}^1 -homotopy theory is introduced by Morel and Voevodsky (1998) as a framework to study cohomology theories in algebraic geometry, by importing tools from algebraic topology
- Idea: use the affine line \mathbb{A}^1 as a substitute of the unit interval to get an algebraic version of the homotopy theory
- Can be used to study cohomology theories such as algebraic K -theory, Chow groups (motivic cohomology) and many others
- Advantage: has many a lot of structures coming from both topological and algebraic geometrical sides

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- Non-commutative geometry and singularity categories (Tabuada, Blanc-Robalo-Toën-Vezzosi)

Some topological background

- A **spectrum** \mathbb{E} is a sequence $(E_n)_{n \in \mathbb{N}}$ of pointed spaces (e.g. CW-complexes or simplicial sets) together with continuous maps $\sigma_n : S^1 \wedge E_n \rightarrow E_{n+1}$ called **suspension maps**

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- From an ∞ -categorical point of view, the category of spectra is the stabilization of the category of spaces, and is the universal stable (triangulated) category

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- **Bigraded \mathbb{A}^1 -homotopy sheaves**: for $X \in \mathbf{H}(S)$, $\mathbb{A}_{a,b}^1(X)$ is the Nisnevich sheaf on Sm_S associated to the presheaf

$$U \mapsto [U \wedge S^{a-b} \wedge \mathbb{G}_m^b; X]_{\mathbf{H}_\bullet(S)}$$

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- $\mathbf{SH}(S)$ is the universal stable ∞ -category which satisfies Nisnevich descent and \mathbb{A}^1 -invariance (Robalo, Drew-Gallauer)

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- The 1-line is also computed (Rondigs-Spitzweck-Østvær):

$$0 \rightarrow K_2^M \rightarrow {}_{n=24} \rightarrow {}_{n+1,n}(\mathbb{1}_k) \rightarrow {}_{n+1,n}f_0(\mathbf{KQ})$$

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- They satisfy formal properties axiomatizing important theorems such as duality, base change and localization.

Thom spaces and relative purity

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Grothendieck's absolute purity conjecture
Motivic homotopy theory
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Orientations

- An **absolute motivic spectrum** is the data of $\mathbb{E}_X \in \mathbf{SH}(X)$ for every scheme X , together with natural isomorphisms $f^* \mathbb{E}_X \simeq \mathbb{E}_Y$ for every morphism $f : Y \rightarrow X$

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- For oriented spectra, Deglise defined fundamental classes using Chern classes

Bivariant groups

- For $f : X \rightarrow S$ be a separated morphism of finite type, $v \in K_0(X)$ and $\mathbb{E} \in \mathbf{SH}(S)$, define the **\mathbb{E} -bivariant groups** (or **Borel-Moore \mathbb{E} -homology**) as

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- If S is a field and $\mathbb{E} = \mathbf{HZ}$, then $\mathbb{E}_i(X=S; v) = CH_r(X; i)$ are the higher Chow groups, where r is the virtual rank of v

Functoriality of bivariant groups

- Base change:

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ g \downarrow & \Delta & \downarrow f \\ T & \xrightarrow{p} & S \end{array}$$

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- Product: if \mathbb{E} has a ring structure, $X \xrightarrow{f} Y \xrightarrow{g} S$

$$\mathbb{E}_m(X=Y; w) \otimes \mathbb{E}_n(Y=S; \nu) \rightarrow \mathbb{E}_{m+n}(X=S; w + f \nu)$$

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- The construction uses the *deformation to the normal cone*

Euler class and excess intersection formula

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- **Motivic Gauss-Bonnet formula** (Levine, Deglise-J.-Khan)
 For $p : X \rightarrow S$ a smooth and proper morphism

$$(X=S) = p_* e(T_p)$$

where $(X=S)$ is the *categorical Euler characteristic*

The absolute purity property

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- From this property Cisinski-Déglise deduce that the rational motivic Eilenberg-Mac Lane spectrum $\mathbf{H}\mathbb{Q}$ also satisfies absolute purity, mainly because $\mathbf{H}\mathbb{Q}$ is a direct summand of $\mathbf{KGL}_{\mathbb{Q}}$ by the Grothendieck-Riemann-Roch theorem

The Main result

Theorem (Deglise-Fasel-J.-Khan):

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- The "switching factors" endomorphism of $\mathbb{P}^1 \wedge \mathbb{P}^1$ induces a decomposition of the sphere spectrum $\mathbb{1}_{\mathbb{Q}}$ into the direct sum of the plus-part $\mathbb{1}_{+, \mathbb{Q}}$ and the minus-part $\mathbb{1}_{-, \mathbb{Q}}$ (Morel)

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- The + -part $\mathbb{1}_{+, \mathbb{Q}}$ agrees with $\mathbf{H}\mathbb{Q}$ (Cisinski-Deglise)
- Therefore it suffices to show that the minus part satisfies absolute purity

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- This proves the absolute purity of $\mathbb{1}_{\mathbb{Q}}$ when 2 is invertible on the base scheme, since \mathbf{KQ} is only well-defined in this case

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- The key lemma then reduces the absolute purity of $\mathbb{1}_{\mathbb{Q}}$ in mixed characteristic to the case of \mathbb{Q} -schemes, which can be proved using Popescu's theorem: a closed immersion of a finite regular schemes over a perfect field is a limit of closed immersions of smooth schemes

Some applications

- Our method can be used to deduce the following new results in mixed characteristic:
 - The six functors preserve constructible objects in the rational stable motivic homotopy category $\mathbf{SH}(\cdot; \mathbb{Q})$
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- Related work: absolute purity of the sphere spectrum over a Dedekind domain (Frankland-Nguyen-Spitzweck, work in progress)

Thank you!